A new proof of the formulas involving the distributions δ^+ and δ^-

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1. Introduction Throughout in this paper, $(\mathcal{O}_{-\alpha})$ will mean for any fixed $\alpha>0$ the linear space of all (C^∞)-functions φ on $\mathbf{R}$ such that $\varphi^{(p)}(t)=O\left(\frac{1}{|t|^\alpha}\right)$ for $p=0, 1, \dots$ (as $|t|\to\infty$). $(\mathcal{O}'_{-\alpha})$ will mean the space of all continuous linear functionals on $(\mathcal{O}_{-\alpha})$. For basic facts concerning the space (O_α) and its dual (O'_α) we refer to [2] and [7]..

The purpose of this note is to give a new proof of the formulas (4) (utilized constantly in quantum mechanics) by a direct and short method, based upon the well known formulas of J. PLEMELJ.

An entirely different technique is described in [2, pp. 60—66], and for other distributional spaces in [1, pp. 155—156], [3, pp. 49—50], [4, pp. 975—976], [5, pp. 426—427], and [9, pp. 85—86].

2. Lemmas. We begin with a lemma on the distribution $\mathrm{Vp}\frac{1}{t}$ and recall a theorem of Plemelj.

First of all let us observe that the linear form $\delta:\varphi\to\varphi(0)$ is continuous on $(\mathcal{O}_{-\alpha})$ since

$$|\langle\delta,\varphi\rangle|\leqq M\max_t\{(1+|t|)^\alpha|\varphi(t)|\}.$$

If φ_n converges in $(\mathcal{O}_{-\alpha})$ to zero as $n\to\infty$, then $\langle\delta,\varphi_n\rangle$ tends to zero. Thus δ is a distribution in $(\mathcal{O}'_{-\alpha})$.

In [2, p. 62] it is proved by means of the distribution δ^+ that $\mathrm{Vp}\frac{1}{t}$ is a distribution in $(\mathcal{O}'_{-\alpha})$. In the following this will be proved directly.

Lemma 1. *The linear form* $\mathrm{Vp}\frac{1}{t}$ *defined by*

$$\left\langle \mathrm{Vp}\frac{1}{t},\varphi\right\rangle=\mathrm{Vp}\int_{-\infty}^{\infty}\frac{\varphi(t)}{t}\,dt=\lim_{\varepsilon\to0}\int_{|t|\geqq\varepsilon}\frac{\varphi(t)}{t}\,dt \tag{1}$$

is a distribution in $(\mathcal{O}'_{-\alpha})$.

Proof. For each $\varphi\in(\mathcal{O}_{-\alpha})$ the limit (1) exists. The argument is the same as in the case of the test functions that belong to the space $(\mathscr{D})$. Observe that the integrand is $O\left(\frac{1}{|t|^{\alpha+1}}\right)$ for large $|t|$. On the other hand, for each $\varepsilon>0$ the linear form

$$\varphi\to\int\limits_{|t|\geqq\varepsilon}\frac{\varphi(t)}{t}\,dt=\left\langle\left(\frac{1}{t}\right)_{\varepsilon},\varphi\right\rangle \tag{2}$$

is a distribution in $(\mathcal{O}'_{-\alpha})$ defined on $\mathbf{R}$. In fact, we can write

$$\left|\left\langle\left(\frac{1}{t}\right)_{\varepsilon},\varphi\right\rangle\right|\leqq 2\int\limits_{\varepsilon}^{\infty}\frac{|\varphi(t)|}{|t|}\,dt\leqq\left(2\int\limits_{\varepsilon}^{\infty}\frac{dt}{|t|(1+|t|)^{\alpha}}\right)\max_{t}\{(1+|t|)^{\alpha}|\varphi(t)|\}.$$

Now suppose that φ_n converges in $(\mathcal{O}_{-\alpha})$ to zero as $n\to\infty$. Then the sequence of numbers $\left\langle\left(\frac{1}{t}\right)_{\varepsilon},\varphi_n\right\rangle$ tends to zero.

By the theorem on the convergence of distributions in $(\mathcal{O}'_{-\alpha})$ it follows that the limit (1) defines a distribution, that is, $\left(\frac{1}{t}\right)_{\varepsilon}$ converges to $\mathrm{Vp}\frac{1}{t}$ in $(\mathcal{O}'_{-\alpha})$ as ε tends to zero.

Lemma 2 (J. Plemelj). *Let f be a function on $\mathbf{R}$ to $\mathbf{C}$ satisfying the (Hölder) condition H on every compact subset of $\mathbf{R}$, and with $f(t)=O\left(\frac{1}{|t|^{\lambda}}\right)$ for large $|t|$ for some $\lambda>0$. If z tends from $D^+=\{z|\operatorname{Im}(z)>0\}$ or from $D^-=\{z|\operatorname{Im}(z)<0\}$ to a point $a\in\mathbf{R}$, then the integral*

$$F(z)=\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{f(t)}{t-z}\,\mathrm{dt}$$

converges to the limits

$$F^{\pm}(a)=\pm\frac{1}{2}f(a)+\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{f(t)}{t-a}\,dt,$$

respectively, where the singular integral is taken as the Cauchy principal value (with respect to the point a).

3. The Theorem. *If*

$$\langle\delta^{\pm},\varphi\rangle=\lim_{\varepsilon\to+0}\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{\varphi(t)}{t\mp i\varepsilon}\,dt \tag{3}$$

then

$$\delta^{\pm}=\pm\frac{\delta}{2}+\frac{1}{2\pi i}\mathrm{Vp}\frac{1}{t}, \tag{4}$$

in the sense of $(\mathcal{O}'_{-\alpha})$.

Proof. First we prove, *independently* of the relations (4), that the linear forms $\delta^\pm$ are distributions in $(\mathcal{O}'_{-\alpha})$.

Note that for each $\varepsilon > 0$ the integrals in (3) converge because the integrands are $O\left(\frac{1}{|t|^{\alpha+1}}\right)$. Also, for each $\varepsilon > 0$, the linear forms

(5) $$\varphi \to \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\varphi(t)}{t \mp i\varepsilon}\, dt$$

are distributions. In fact, identifying the distributions with the functions

$$t \to \frac{1}{t \mp i\varepsilon}$$

to which they correspond, we have

$$\left| \left\langle \frac{1}{t \pm i\varepsilon}, \varphi \right\rangle \right| \leqq \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dt}{(1+|t|)^\alpha \sqrt{t^2+\varepsilon^2}} \right) \max_t \{(1+|t|)^\alpha |\varphi(t)|\}.$$

The integral being convergent, the rest of the argument is obvious from what has been shown in Lemma 1.

Now let us consider the integral of the Cauchy type

$$\hat{\varphi}(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\varphi(t)}{t-z}\, dt, \quad \operatorname{Im}(z) \neq 0.$$

Note that $\hat{\varphi}$ is holomorphic in $D^+ = \{z | z = x + i\varepsilon\}$ $(\varepsilon > 0)$ and in $D^- = \{z | z = x - i\varepsilon\}$ $(\varepsilon > 0)$. Every function $\varphi \in (\mathcal{O}_{-\alpha})$ is bounded on $\mathbf{R}$ and, being a (C^∞)-function, satisfies with each of its derivatives condition H on every compact subset of $\mathbf{R}$. The range of the distributions (5) coincides with the range of the function $\hat{\varphi}$ for $z = \pm i\varepsilon$, respectively. The limits (3) are equal with the limits of $\hat{\varphi}(z)$ as z approaches to the point $a = 0$ along the imaginary axis from D^+ and D^-, respectively. By Lemma 2 the limits

$$\langle \delta^\pm, \varphi \rangle = \lim_{\varepsilon \to +0} \frac{1}{2\pi i} \left\langle \frac{1}{t \mp i\varepsilon}, \varphi \right\rangle = \lim_{\varepsilon \to +0} \hat{\varphi}(\pm i\varepsilon)$$

exist for every $\varphi \in (\mathcal{O}_{-\alpha})$. Applying the theorem on the convergence of distributions, it follows that δ^+ and δ^- are actually distributions in $(\mathcal{O}'_{-\alpha})$.

At the same time we have

$$\langle \delta^\pm, \varphi \rangle = \pm \frac{\varphi(0)}{2} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\varphi(t)}{t}\, dt = \pm \frac{\langle \delta, \varphi \rangle}{2} + \frac{1}{2\pi i} \left\langle \mathrm{Vp}\, \frac{1}{t}, \varphi \right\rangle.$$

This implies the relations (4). The proof is complete.

Remark 1. Let $\delta_{(a)}$ be a distribution defined by $\langle \delta_{(a)}, \varphi \rangle = \varphi(a)$, $a \in \mathbf{R}$ (for $a = 0$, $\delta_{(a)} = \delta$). Let $\mathrm{Vp}\,\frac{1}{t-a}$, $\delta^+_{(a)}$, $\delta^-_{(a)}$ be the distributions deduced from (1) and (3) if in place of the terms t, $t-i\varepsilon$, $t+i\varepsilon$ we set $t-a$, $t-a-i\varepsilon$, $t-a+i\varepsilon$, respectively. In this case, the same method gives

$$\delta^{\pm}_{(a)} = \pm \frac{\delta_{(a)}}{2} + \frac{1}{2\pi i}\,\mathrm{Vp}\,\frac{1}{t-a}.$$

Remark 2. Since Plemelj's theorem is valid in a complex Banach space ([6]), it is possible to derive the same formulas if $\delta^{\pm}$ are vector-valued distributions (compare with [8, pp. 659—661]).

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